J. DIFFERENTIAL GEOMETRY
14 (1979) 237–240

# REAL 4-DIMENSIONAL KÄHLERIAN MANIFOLDS OF CONSTANT SCALAR CURVATURE

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## 1. Introduction

We denote by $(M, g)$ a compact oriented Riemannian manifold of dimension 4 with metric $g$, and by $(M, g, J)$ a compact Kählerian manifold of real dimension 4 with almost complex structure tensor $J$ and Kählerian metric $g$. Let sign $(M)$ and $\chi(M)$ be the sign ature and the Euler-Poincaré characteristic of $M$. It is well known that $\chi(M) \geqslant 0$ holds for an Einstein $(M, g)$ (M. Berger [1]). With respect to the relation between sign $(M)$ and $\chi(M)$, N. Hitchin [4] showed

$$3|\text{sign}\,(M)| \leqslant 2\chi(M) \tag{1.1}$$

for an Einstein $(M, g)$. Later H. Donnelly [3] showed

$$-2\chi(M) \leqslant 3\,\text{sign}\,(M) \leqslant \chi(M) \tag{1.2}$$

for an Einstein $(M, g, J)$. Here the equality of the second inequality of (1.2) holds if and only if $(M, g, J)$ is of constant holomorphic sectional curvature. This second inequality is decomposed as

$$3\,\text{sign}\,(M) \leqslant \frac{1}{96}\pi^{-2}S^2\,\text{Vol}\,(M) \leqslant \chi(M), \tag{1.3}$$

where Vol $(M)$ and $S$ denote the volume and the scalar curvature of $(M, g, J)$, respectively (cf. Remark 5).

Generalizing (1.3) for $(M, g, J)$ of constant scalar curvature, we obtain Theorems A, B and C.

Let $S^2(K)$ and $H^2(-K)$ be a Euclidean 2-sphere of constant curvature $K$ and a hyperbolic 2-space of constant curvature $-K$, with the natural Kählerian structures.

**Theorem A.** *If a compact Kählerian manifold $(M, g, J)$ of real dimension 4 has constant scalar curvature $S$, then*

$$3\,\text{sign}\,(M) \leqslant \frac{1}{96}\pi^{-2}S^2\,\text{Vol}\,(M) \tag{1.4}$$

Received April 21, 1977.

*holds. If the equality of* (1.4) *holds, then either*

(i) $(M, g, J)$ *is of constant holomorphic sectional curvature, or*

(ii) $(M, g, J)$ *is locally a product* $S^2(K) \times H^2(-K)$.

For the case $S > 0$, we have a characterization of a complex projective 2-space.

**Theorem B.** *Let* $(M, g, J)$ *be a real 4-dimensional compact Kählerian manifold with positive constant scalar curvature* $S$. *Then we have inequality* (1.4), *and the equality holds if and only if* $(M, g, J)$ *is holomorphically isometric to a complex projective 2-space* $(CP^2, g_0, J_0)$ *with Fubini-Study metric* $g_0$ *of constant holomorphic sectional curvature* $H = S/6$.

By $CE^2$ we denote a complex Euclidean 2-space.

**Theorem C.** *In a compact Kählerian manifold* $(M, g, J)$ *of real dimension* 4, *if the scalar curvature* $S = 0$ *and* sign $(M) = 0$, *then* $(M, g, J)$ *is one of the following spaces;*

$$CE^2/\Gamma_1, \ S^2(K) \times H^2(-K)/\Gamma_2,$$

*where* $\Gamma_1$ *or* $\Gamma_2$ *denotes some discrete subgroup of the automorphism group of* $CE^2$ *or* $S^2(K) \times H^2(-K)$.

## 2. Preliminaries

By $R = (R^i{}_{jkl})$, $Ric = (R_{jl} = R^i{}_{jil})$ and $S = (g^{ji}R_{ji})$ we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature of a Riemannian manifold $(M, g)$. The first Pontrjagin form $p_1$ of a 4-dimensional compact oriented Riemannian manifold $(M, g)$ is given by

$$p_1 = \frac{1}{4}\pi^{-2}(R_{rs12}R^{rs34} - R_{rs13}R^{rs24} + R_{rs14}R^{rs23})\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4,$$

where $(x^i)$ denotes a local coordinate system which is positively related to the given orientation of $M$. The sign ature of $M$ is given by

$$3 \text{ sign } (M) = \int_M p_1.$$

Now let $(M, g, J)$ be a compact Kählerian manifold of real dimension 4. The orientation of $M$ is defined by considering $(e_1, Je_1, e_2, Je_2)$ as a positive frame. Then the sign ature of $(M, g, J)$ is expressed as

$$-3 \text{ sign } (M) = \frac{1}{16}\pi^{-2}\int_M (|R|^2 - 2|\text{Ric}|^2)\, dM \tag{2.1}$$

by a result of H. Donnelly [3], where $dM$ denotes the volume element of $(M, g, J)$, and

$$|R|^2 = R_{ijkl}R^{ijkl}, \ |\text{Ric}|^2 = R_{ji}lR^{ji}.$$

To get (2.1) H. Donnelly applied his invariance theorem [2] to Kählerian manifolds. Namely, denoting by $*$ the Hodge star operator associated with the Kählerian structure, the function $*p_1$ is an invariant of order 4 in his sense.

**Remark 1.** In the case of a compact oriented Riemannian manifold $(M, g)$ of dimension 4, denote by $*$ the Hodge star operator associated with the Riemannian structure. Then $*p_1$ is of $SO(4)$-invariant type and not of $O(4)$-invariant type, because $*p_1$ contains the so-called determinant part.

**Remark 2.** (2.1) is also directly proved. Let $x$ be an arbitrary point of $(M, g, J)$ and $(e_1, Je_1, e_2, Je_2)$ an orthonormal frame at $x$. This frame may be chosen so that the Ricci tensor is diagonal. Then (2.1) is verified from the local expression of $p_1$.

**Remark 3.** Denote the Bochner curvature tensor of $(M, g, J)$ by $B = (B^i{}_{jkl})$, define a (0, 2)-tensor $G$ by

$$G_{ji} = R_{ji} - \tfrac{1}{4} S g_{ji},$$

and put $|B|^2 = B_{ijkl}B^{ijkl}$ and $|G|^2 = G_{ji}G^{ji}$. Then $G = 0$ holds on $M$ if and only if $(M, g)$ is an Einstein manifold. Furthermore, it is known that as far as curvature tensor norms are concerned the condition $|B| = |G| = 0$ is most effective to the conclusion of constancy of holomorphic sectional curvature (cf. S. Tanno [6, Theorem 4.3]).$|B|$ is given by (S. Tanno [6], [7])

$$|B|^2 = |R|^2 - 2|\mathrm{Ric}|^{12} + \frac{1}{6} S^2. \tag{2.2}$$

## 3. Proofs of Theorems A, B, and C

By (2.1) and (2.2) we obtain

$$48\pi^2 \operatorname{sign}(M) = -\int_M |B|^2\, dM + \frac{1}{6}\int_M S^2\, dM. \tag{3.1}$$

Hence we obtain

**Proposition 3.1.** *Let $(M, g, J)$ be a compact Kählerian manifold of real dimension 4. Then*

$$288\,\pi^2 \operatorname{sign}(M) \leqslant \int_M S^2\, dM, \tag{3.2}$$

*where the equality holds if and only if $B = 0$.*

*Proof of Theorem A.* If we assume that the scalar curvature $S$ is constant, then (1.4) follows from (3.2). The equality holds if and only if $B = 0$. Therefore to complete the proof of Theorem A it suffices to apply the following.

**Proposition 3.2** (M. MATSUMOTO & S. TANNO [5]). *If a Kählerian manifold $(M, g, J)$ has vanishing Bochner curvature tensor $B$ and constant scalar curvature $S$, then either*

*(i) $(M, g, J)$ is of constant holomorphic sectional curvature, or*

*(ii) $(M, g, J)$ is locally a product of two spaces of constant holomorphic sectional curvature $K > 0$ and $-K$.*

**Remark 4.** For $(M, g, J) = (CP^2, g_0, J_0)$ with constant holomorphic sectional curvature $H = 4$, we have Vol $(M) = \frac{1}{2}\pi^2$, sign $(M) = 1$, and $S = 6H = 24$.

Let $N^2(-K)$ be a compact oriented Riemann surface of genus $> 2$ with constant curvature $-K < 0$. Then $(M, g, J) = S^2(K) \times N^2(-K)$ has the scalar curvature $S = 0$ and sign $(M) = 0$.

**Remark 5.** If $(M, g, J)$ is a compact Einstein Kählerian manifold of real dimension 4, then

$$\frac{1}{96}\pi^{-2}S^2 \,\mathrm{Vol}\,(M) \leqslant \chi(M), \tag{3.3}$$

where the equality holds if and only if $(M, g, J)$ is of constant holomorphic sectional curvature (S. Tanno [6]). (1.4) and (3.3) imply the second inequality of (1.2), i.e., (1.3).

*Proof of Theorem* B. Since $S^2(K) \times H^2(-K)$ has vanishing scalar curvature, $(M, g, J)$ is of constant holomorphic sectional curvature $H > 0$ by Theorem A and $S > 0$. Generally we see that a complete Kählerian manifold $(M^{2n}, g, J)$ of positive constant holomorphic sectional curvature is simply connected and hence holomorphically isometric to $(CP^n, g_0, J_0)$. Consequently, our $(M, g, J)$ is holomorphically isometric to $(CP^2, g_0, J_0)$ of constant holomorphic sectional curvature $H = S/6$.

Proof of Theorem C is easy.

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